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Renormalization of $\Delta B = 2$ Transitions in the Static Limit Beyond Leading Logarithms

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Abstract

The renormalization group evolution of $\Delta B = 2$ transition operators is studied at leading order in heavy quark effective theory and at next-to-leading order in QCD. We calculate new contributions that were not taken into account in previous work and obtain the complete result to next-to-leading logarithmic accuracy. The various scheme and scale dependences and their cancellation in physical quantities are discussed explicitly.

The phenomenon of $B_d - \bar{B}_d$ mixing is one of the most interesting and important effects in flavordynamics. Most notably it is sensitive to the mass and the weak coupling V_{td} of the top quark, which play an important role in testing the standard model picture of quark mixing in general and of the unitarity triangle and its implications for CP violation in particular. Unlike many other loop-induced FCNC processes, $B_d - \bar{B}_d$ mixing has been observed and its measurement provides some constraint on standard model parameters already today. As our knowledge of the top quark mass increases and the experimental result for the mixing parameter $x_d = \Delta m_{B_d}/\Gamma_B$ gains in accuracy, $B - \bar{B}$ mixing has the potential of providing very valuable information on $|V_{td}|$. Such information is crucial for a decisive test of our current understanding of flavordynamics.

Unfortunately the analysis and interpretation of $B - \bar{B}$ mixing is limited on the theoretical side through the need to evaluate the hadronic matrix element of a four-quark operator between the B meson states. This computation involves QCD long distance dynamics, which pose a notoriously difficult problem not yet solved satisfactorily to date. Ultimately the required matrix element should be calculable in the framework of lattice gauge theory. However, since the b-quark mass is larger than currently attainable lattice cutoffs, it is difficult to treat B mesons with fully dynamical quarks in numerical simulations. In this situation it is useful to employ heavy quark effective theory (HQET). This approach allows one to extract the dependence of the matrix element on the heavy mass m_b analytically, leaving only light degrees of freedom as dynamical variables in a lattice calculation.

It is the purpose of this Letter to discuss the relation of the full theory operator to its HQET counterparts to leading order in the $1/m$ expansion and to next-to-leading order in renormalization group improved perturbation theory, thereby generalizing, respectively completing results that have already been obtained in the literature [1, 2]

The effective hamiltonian describing $B_d - \bar{B}_d$ mixing can be written as

$$\mathcal{H}_{eff}^{\Delta B=2} = \frac{G_F^2}{16\pi^2} M_W^2 (V_{tb}^* V_{td})^2 C(x_t(\mu_t), \mu_t, \mu_b) Q + h.c. \quad (1)$$

with

$$Q = (\bar{b}d)_{V-A}(\bar{b}d)_{V-A} \quad (2)$$

and $x_t = m_t^2/M_W^2$. From this hamiltonian the $B_d - \bar{B}_d$ mixing parameter $x_d = \Delta m_{B_d}/\Gamma_B$ can be derived with the result

$$x_d = \tau_B \frac{G_F^2}{16\pi^2} \frac{M_W^2}{m_B} |V_{td}|^2 C(x_t(\mu_t), \mu_t, \mu_b) \langle Q(\mu_b) \rangle \quad (3)$$

where

$$\langle Q(\mu_b) \rangle \equiv \langle \bar{B} | Q(\mu_b) | B \rangle = \frac{8}{3} f_B^2 m_B^2 B_B(\mu_b) \quad (4)$$

is the required hadronic matrix element mentioned above. f_B is the B-meson decay constant in the normalization in which $f_\pi = 131 MeV$ and B_B is the bag parameter, defined through (4).

The effective hamiltonian formalism is based on the operator product expansion, which achieves a factorization of short-distance and long-distance contributions to the amplitude in (3). The dependence on the top-quark mass and the short-distance QCD effects from the

high matching scale $\mu_t = \mathcal{O}(m_t) = \mathcal{O}(M_W)$, at which top and W are integrated out, down to a scale $\mu_b = \mathcal{O}(m_b)$ are contained in the Wilson coefficient function $C(x_t(\mu_t), \mu_t, \mu_b)$. This coefficient is calculable in RG improved perturbation theory. It is independent of the scale μ_t to the considered order in α_s . On the other hand C does depend on the low scale μ_b and on the renormalization scheme chosen to subtract divergences in its calculation. Both dependences are canceled by the hadronic matrix element in the product $C \cdot \langle Q \rangle$. The important feature of the scale and scheme dependence and their cancellation appears only at $\mathcal{O}(\alpha_s)$ and hence can be meaningfully addressed only beyond the leading logarithmic approximation. For the coefficient function $C(x_t(\mu_t), \mu_t, \mu_b)$ the next-to-leading order analysis has been performed and discussed in detail in [3]. This calculation resums the leading ($\alpha_s^n \ln^n \mu_t/\mu_b$) and next-to-leading ($\alpha_s^{n+1} \ln^n \mu_t/\mu_b$) logarithmic QCD corrections to all orders $n = 0, 1, 2, \dots$ in perturbation theory and eliminates the μ_t -dependence to order $\mathcal{O}(\alpha_s)$, which considerably reduces the theoretical uncertainty in the short-distance calculation. In fact the μ_t -dependence is entirely negligible after the inclusion of NLO corrections [3].

Whereas the coefficient C incorporates the short-distance QCD effects from scale μ_t down to μ_b , the matrix element $\langle Q \rangle$ contains the "long-distance" contributions below $\mu_b = \mathcal{O}(m_b)$. However, since QCD continues to be perturbative from scales μ_b down to $\mu \sim 1\text{GeV}$, the corresponding contributions to the matrix element may still be extracted in an analytical way using the framework of HQET. Within this formalism the full theory matrix element $\langle Q(\mu_b) \rangle$ may be expanded as follows [1]

$$\langle Q(\mu_b) \rangle = C_1(\mu_b, \mu) \langle \tilde{Q}(\mu) \rangle + C_2(\mu_b, \mu) \langle \tilde{Q}_s(\mu) \rangle + \mathcal{O}\left(\frac{1}{m}\right) \quad (5)$$

where we are working to leading order in $1/m$. In this order two operators appear in the effective theory. They are given by

$$\tilde{Q} = 2(\bar{h}^{(+)}d)_{V-A}(\bar{h}^{(-)}d)_{V-A} \quad \tilde{Q}_s = 2(\bar{h}^{(+)}d)_{S-P}(\bar{h}^{(-)}d)_{S-P} \quad (6)$$

with $(\bar{h}d)_{V-A} \equiv \bar{h}\gamma^\mu(1 - \gamma_5)d$ and $(\bar{h}d)_{S-P} \equiv \bar{h}(1 - \gamma_5)d$. Here the heavy quark fields are defined in the usual way as

$$h^{(\pm)}(x) = e^{\pm imv \cdot x} \frac{1 \pm \not{v}}{2} b(x) \quad (7)$$

\tilde{Q} , \tilde{Q}_s are the HQET operators relevant for the case of a $B \rightarrow \bar{B}$ transition. The field $\bar{h}^{(+)}$ creates a heavy quark, while $\bar{h}^{(-)}$ annihilates a heavy antiquark. Since the effective theory field $\bar{h}^{(+)}$ ($\bar{h}^{(-)}$) cannot, unlike the full theory field \bar{b} in Q (2), at the same time annihilate (create) the heavy antiquark (heavy quark), explicit factors of two have to appear in (6). Due to the mass of the b quark an additional operator \tilde{Q}_s with scalar–pseudoscalar structure is generated in the effective theory.

(5) has the form of an operator product expansion. The short-distance contributions of scales between μ_b and μ are factorized into the coefficients $C_{1,2}$, while $\langle \tilde{Q}(\mu) \rangle$ and $\langle \tilde{Q}_s(\mu) \rangle$ contain the contributions below μ . In particular the leading dependence of $\langle Q(\mu) \rangle$ on the b -quark mass is fully separated into the coefficient functions and no reference to m_b is present in the effective theory matrix elements, which makes them more readily computable by lattice methods than $\langle Q(\mu) \rangle$.

In a first step the coefficients $C_{1,2}$ have to be determined at $\mu = \mu_b$ by calculating the matching

of the full theory operator Q onto its HQET counterparts \tilde{Q} , \tilde{Q}_s to $\mathcal{O}(\alpha_s)$ in perturbation theory. After performing a proper factorization of long- and short-distance contributions, the coefficient functions are independent of the infrared properties of the amplitude (5), in particular of the treatment of external lines. Therefore, for the determination of the coefficients, the arbitrary external states may be chosen according to calculational convenience. The simplest technique is to put the heavy quark lines on shell, to set all other external momenta to zero and to regularize both ultraviolet and infrared divergences dimensionally. In this case the $\mathcal{O}(\alpha_s)$ corrections to the matrix elements of \tilde{Q} , \tilde{Q}_s vanish. Using this method, performing the necessary renormalizations in the \overline{MS} scheme with anticommuting γ_5 (NDR scheme) and subtracting evanescent contributions in the usual way [4], we obtain

$$C_1(\mu_b, \mu_b) = 1 + \frac{\alpha_s(\mu_b)}{4\pi} \left[(\tilde{\gamma}_{11}^{(0)} - \gamma^{(0)}) \ln \frac{\mu_b}{m_b} + \tilde{B} - B \right] \quad (8)$$

$$C_2(\mu_b, \mu_b) = \frac{\alpha_s(\mu_b)}{4\pi} \tilde{B}_s \quad (9)$$

where

$$\gamma^{(0)} = 6 \frac{N-1}{N} \quad \tilde{\gamma}_{11}^{(0)} = -6C_F \quad C_F = \frac{N^2-1}{2N} \quad (10)$$

$$\tilde{B} - B = -\frac{8N^2 + 9N - 15}{2N} \quad B = 11 \frac{N-1}{2N} \quad \tilde{B}_s = -2(N+1) \quad (11)$$

and N is the number of colors. For $N = 3$ this result coincides with the calculation of [1]. The logarithmic term in (8) reflects the $\mathcal{O}(\alpha_s)$ scale dependence of the matrix elements of Q and \tilde{Q} . Accordingly its coefficient is given by the difference in the one-loop anomalous dimensions of these operators, $\gamma^{(0)}$ and $\tilde{\gamma}_{11}^{(0)}$. At the same time this logarithm expresses the complete dependence of the full theory matrix element $\langle Q \rangle$ on the b -quark mass, to $\mathcal{O}(\alpha_s)$ and to leading order in $1/m$, which is thus factorized into the short-distance coefficient function C_1 . Besides the logarithm (8) contains a scheme dependent constant. We have written this term in the form $\tilde{B} - B$ in order to make the cancellation of scheme dependences, to be discussed below, more transparent. Here B is a constant characterizing the scheme dependence of the full theory matrix element $\langle Q \rangle$, given here in the NDR scheme (see also [3]).

In a next step we want to evolve the coefficients $C_{1,2}$ from the b -scale $\mu_b = \mathcal{O}(m_b)$ down to a typical hadronic scale $\mu \sim 1\text{GeV}$ in next-to-leading order of RG improved perturbation theory. In this way we are resumming to all orders the leading ($\alpha_s^n \ln^n \mu_b/\mu$) and next-to-leading ($\alpha_s^{n+1} \ln^n \mu_b/\mu$) logarithmic contributions, which are of $\mathcal{O}(1)$ and $\mathcal{O}(\alpha_s)$, respectively, in the limit $\mu_b \gg \mu$, where the logarithms become large.

The evolution of the coefficients $\vec{C}^T = (C_1, C_2)$ is described by the RG equation

$$\frac{d}{d \ln \mu} \vec{C}(\mu_b, \mu) = \tilde{\gamma}^T \vec{C}(\mu_b, \mu) \quad (12)$$

where the matrix of anomalous dimensions of the operators \tilde{Q} , \tilde{Q}_s can be written to two-loop accuracy as

$$\tilde{\gamma} = \frac{\alpha_s}{4\pi} \begin{pmatrix} \tilde{\gamma}_{11}^{(0)} & 0 \\ \tilde{\gamma}_{21}^{(0)} & \tilde{\gamma}_{22}^{(0)} \end{pmatrix} + \left(\frac{\alpha_s}{4\pi} \right)^2 \begin{pmatrix} \tilde{\gamma}_{11}^{(1)} & 0 \\ \tilde{\gamma}_{21}^{(1)} & \tilde{\gamma}_{22}^{(1)} \end{pmatrix} \quad (13)$$

The first row of $\tilde{\gamma}$ has already been considered to next-to-leading order in [2] and we will use the results obtained in this paper here. The vanishing of the (12)-entry in $\tilde{\gamma}$ means that \tilde{Q} does not mix into the operator \tilde{Q}_s . The second row of $\tilde{\gamma}$ will be the subject of the present article.

Using the well known general expressions for the solution of (12) at next-to-leading order [5], the structure of $\tilde{\gamma}$ given in (13) and the initial values of $C_{1,2}$ from (8), (9) we obtain after inserting the solution $\vec{C}(\mu_b, \mu)$ into (5)

$$\begin{aligned} \langle Q(\mu_b) \rangle = & \left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu)} \right]^{\tilde{d}_1} \left(1 + \frac{\alpha_s(\mu_b)}{4\pi} \left[(\tilde{\gamma}_{11}^{(0)} - \gamma^{(0)}) \ln \frac{\mu_b}{m_b} + \tilde{B} - B - \tilde{J} \right] + \frac{\alpha_s(\mu)}{4\pi} \tilde{J} \right) \langle \tilde{Q}(\mu) \rangle \\ & + \frac{\alpha_s(\mu_b)}{4\pi} \frac{\tilde{\gamma}_{21}^{(0)}}{\tilde{\gamma}_{11}^{(0)} - \tilde{\gamma}_{22}^{(0)}} \tilde{B}_s \left(\left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu)} \right]^{\tilde{d}_1} - \left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu)} \right]^{\tilde{d}_2} \right) \langle \tilde{Q}(\mu) \rangle \\ & + \frac{\alpha_s(\mu_b)}{4\pi} \left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu)} \right]^{\tilde{d}_2} \tilde{B}_s \langle \tilde{Q}_s(\mu) \rangle \end{aligned} \quad (14)$$

Here

$$\tilde{d}_1 = \frac{\tilde{\gamma}_{11}^{(0)}}{2\beta_0} \quad \tilde{d}_2 = \frac{\tilde{\gamma}_{22}^{(0)}}{2\beta_0} \quad \tilde{J} = \frac{\tilde{d}_1}{\beta_0} \beta_1 - \frac{\tilde{\gamma}_{11}^{(1)}}{2\beta_0} \quad (15)$$

where

$$\beta_0 = \frac{11N - 2f}{3} \quad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nf - 2C_F f \quad (16)$$

are the beta-function coefficients with N (f) the number of colors (flavors).

Eq. (14) represents the relation between the matrix element of the full theory operator Q , normalized at the "high" scale μ_b , and the matrix elements of the leading HQET operators \tilde{Q} and \tilde{Q}_s , normalized at a low scale $\mu \sim 1\text{GeV}$. It is valid to leading order in the $1/m_b$ expansion and to next-to-leading order in RG improved QCD perturbation theory. This leading contribution in the heavy quark expansion corresponds to the static limit for the b -quark. Going beyond this approximation requires the consideration of several new operators, which arise at the next order in $1/m$. These contributions have been studied in [6] in the leading logarithmic approximation.

We further remark that (14) as it stands is valid in the continuum theory. In order to use lattice results one still has to perform an $\mathcal{O}(\alpha_s)$ matching of \tilde{Q} to its lattice counterpart. This step however does not involve any further renormalization group improvement since by means of (14) $\langle \tilde{Q} \rangle$ is already normalized at the appropriate low scale μ . For \tilde{Q}_s , whose coefficient is $\mathcal{O}(\alpha_s)$, a tree level matching is sufficient.

Note that due to the particular form of the RG evolution and the fact that C_2 is only $\mathcal{O}(\alpha_s)$, the anomalous dimension coefficients $\tilde{\gamma}_{21}^{(1)}$ and $\tilde{\gamma}_{22}^{(1)}$ in (13) do not appear in (14) and hence are irrelevant at NLO. The one-loop anomalous dimension $\tilde{\gamma}_{11}^{(0)}$, describing the leading logarithmic evolution of the effective theory operator matrix element $\langle \tilde{Q} \rangle$, has been first calculated in [7] and [8] with the result quoted in (10). The computation of the two-loop anomalous dimension was performed in [2], who obtains in the NDR scheme

$$\tilde{\gamma}_{11}^{(1)} = -\frac{N-1}{12N} \left[127N^2 + 143N + 63 - \frac{57}{N} + \left(8N^2 - 16N + \frac{32}{N} \right) \pi^2 - (28N + 44)f \right] \quad (17)$$

As (14) shows, the complete NLO expression requires in addition to $\tilde{\gamma}_{11}^{(0)}$ and $\tilde{\gamma}_{11}^{(1)}$ the one-loop anomalous dimensions $\tilde{\gamma}_{22}^{(0)}$ and $\tilde{\gamma}_{21}^{(0)}$, which describe the evolution of the operator \tilde{Q}_s and the mixing of \tilde{Q}_s into \tilde{Q} , respectively. We have calculated these quantities and find

$$\tilde{\gamma}_{21}^{(0)} = \frac{N+1}{N} \quad \tilde{\gamma}_{22}^{(0)} = -\frac{3N^2 - 4N - 7}{N} \quad (18)$$

The determination of $\tilde{\gamma}_{21}^{(0)}$ and $\tilde{\gamma}_{22}^{(0)}$ is straightforward (for general reviews see [9, 10]). We only mention that in order to reexpress additional operator structures, arising in the calculation, in terms of the basis operators \tilde{Q} , \tilde{Q}_s , we made use of the identity (valid in four dimensions for $v^2 = 1$)

$$\not{p}\gamma^\mu\gamma^\nu(1-\gamma_5)\otimes\not{p}\gamma_\mu\gamma_\nu(1-\gamma_5)\equiv 4\gamma^\nu(1-\gamma_5)\otimes\gamma_\nu(1-\gamma_5) \quad (19)$$

and of the projection property $\not{p}h^{(\pm)} = \pm h^{(\pm)}$ of the heavy quark effective fields (7). The relation (19) is also necessary for the matching calculation leading to (8), (9).

Let us discuss the structure of relation (14) in more detail. In particular we would like to consider the scale and scheme dependences present in (14) and to clarify their cancellation in physical quantities. The dependence on μ in the first factor on the r.h.s. of (14) is canceled by the μ -dependence of $\langle\tilde{Q}(\mu)\rangle$. The dependence on μ_b of this factor is canceled by the explicit $\ln\mu_b$ term proportional to $\tilde{\gamma}_{11}^{(0)}$. Hence the only scale dependence remaining on the r.h.s., to the considered order $\mathcal{O}(\alpha_s)$, is the one $\sim\alpha_s(\mu_b)\gamma^{(0)}\ln\mu_b$. This is precisely the scale dependence of the full theory matrix element on the l.h.s., which is required to cancel the corresponding dependence of the Wilson coefficient in (3). Similarly the term $\sim\alpha_s(\mu_b)B$ represents the correct scheme dependence of $\langle Q(\mu_b)\rangle$, while the scheme dependence of $\alpha_s(\mu)\tilde{J}$ cancels with the one of $\langle\tilde{Q}(\mu)\rangle$ and the difference $\tilde{B}-\tilde{J}$ is scheme independent by itself. Likewise the remaining contributions involving \tilde{B}_s are scale and scheme independent to the considered order.

The relation (14) involves two different renormalization scales, the scale μ at which the HQET matrix elements are defined and the scale μ_b where the transition from full QCD to HQET is made. As is evident from our discussion, the precise values of these scales are arbitrary as long as $\mu = \mathcal{O}(1\text{GeV})$ and $\mu_b = \mathcal{O}(m_b)$, where the hierarchy $\mu \ll \mu_b$ is assumed. However, since the logarithms $\ln\mu_b/\mu$ are not really very large in the case at hand, one might consider neglecting higher order resummations of logarithms altogether and content oneself with the strict α_s -corrections alone. This limit can be obtained by expanding (14) to first order in α_s . In this case the expression (14) simplifies to the result used in [1] when μ_b is set equal to μ . This approximation is fully consistent. Yet it is useful and interesting to have the more complete expression (14) at hand, which allows one to quantify the effects of the leading and next-to-leading logarithms $\ln\mu_b/\mu$ in this relation.

Numerically, for $\mu_b = 4.8\text{GeV}$, $\mu = 1\text{GeV}$, $\Lambda_{\overline{MS}}^{(4)} = 0.3\text{GeV}$ and $f = 4$ flavors, the coefficient of $\alpha_s(\mu_b)/(4\pi)\tilde{B}_s\langle\tilde{Q}_s(\mu)\rangle$ in (14) reads 1.13, instead of 1 if the summation of logarithms was neglected. The new contribution proportional to \tilde{B}_s to the coefficient of $\langle\tilde{Q}(\mu)\rangle$ is about +0.01. Although these effects are not very large, they have to be taken into account for consistency in a complete analysis within the next-to-leading logarithmic approximation.

Note added: After completion of this work we became aware of reference [11] in which the same topic is addressed. The results of [11] are in agreement with our findings.

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